Jerry R. Hobbs’s Programme and the Heterological Paradox

András Simonyi, Applied Logic Laboratory

MaSzAT, 31 January 2013

Degrees of Reification

Peter runs.

(i) $\text{run}(p)$

(ii) $\exists e(\text{run}'(e, p))$

(iii) $\exists e(\text{run}'(e, p) \wedge \text{Rexist}(e))$

Tom belives that John walks to the pub and Kate runs to the shop.

(iv) $\exists e_1, e_2, e_3, e_4, t, j, k, p, s(\text{tom}(t) \wedge \text{believe}'(e_1, t, e_2) \wedge \text{Present}(e_1) \wedge \text{Rexist}(e_1) \wedge \text{And}'(e_2, e_3, e_4) \wedge \text{john}(j) \wedge \text{walk}'(e_3, j) \wedge \text{Present}(e_3) \wedge \text{To}(e_3, p) \wedge \text{pub}(p) \wedge \text{kate}(k) \wedge \text{run}'(e_4, k) \wedge \text{Present}(e_4) \wedge \text{To}(e_4, s) \wedge \text{shop}(s))$

Ontologically promiscuous logical form

- Standard first-order representation
- *All* morphemes correspond to predications
- The logical form is an instance of the schema
  $\exists \epsilon_1, \ldots, \epsilon_n (\Pi_1(\eta_1^1, \ldots, \eta_{i_1}^1) \wedge \ldots \wedge \Pi_m(\eta_1^m, \ldots, \eta_{i_m}^m))$

Motivation

- Closeness to English (for easy translation)
- Syntactical simplicity
- To treat everything that can be referred to anaphorically as first-class individuals

Expanding the ontology

Axiom schema of plenitude:

$$\forall x_1, \ldots, x_n \exists e\, \Pi'(e, x_1, \ldots, x_n)$$

Axiom schema of real existence:

$$\forall x_1, \ldots, x_n (\Pi(x_1, \ldots, x_n) \leftrightarrow \exists e(\text{Rexist}(e) \wedge \Pi'(e, x_1, \ldots, x_n)))$$

Nonstandard elements in the ontology

- Eventualities (even conjunctive, universal etc.)
- Merely possible entities
- Fictional entities
- Sets/classes and their typical elements
- Concepts

Interpretation as abduction

"Interpretation is the minimal explanation ⟦on the basis of mutual knowledge⟧ of why the text would be true.

. . . .

To interpret a sentence:

Prove the logical form of the sentence,
together with constraints that predicates impose on their arguments,
allowing for coercion,
Merging redundancies where possible,
Making assumptions where necessary."

(Hobbs et al., "Interpretation as abduction", 1993)

Example: Anaphora resolution

I bought a car. The vehicle is perfect.

I bought a car.

Logical form:

$$\exists x, e, c(\text{Ego}(x) \wedge \text{buy}'(e, x, c) \wedge \text{Past}(e) \wedge \text{car}(c))$$

Assumptions:

$$\text{Ego}(I_1), \text{buy}'(E_1, I_1, I_2), \text{Past}(E_1), \text{car}(I_2)$$

The vehicle is perfect.

Logical form:

$$\exists e, c(\text{Present}(e) \wedge \text{perfect}'(e, c) \wedge \text{vehicle}(c))$$

From the background knowledge base:

$$\forall x(\text{car}(x) \rightarrow \text{vehicle}(x))$$

Assumptions:

$$\text{Present}(E_2), \text{perfect}'(E_2, I_2)$$

Weighted abduction

- Assumptions should be checked for consistency
- Conjuncts in the logical form are given assumability costs, e.g.:

  $$\exists e, x(\text{flies}'(e, x)^{\$10} \wedge \text{animal}(x)^{\$20})$$

- Axioms are weighted, e.g.:

  $$\forall x(\text{bird}(x)^{0.8} \wedge \text{etc}_1(x)^{0.3} \longrightarrow \text{flies}(x))$$

  Assuming $\text{etc}_1(I_1)$ to deduce $\text{fly}(I_1)^{\$10}$ would cost $0.3 \times \$10 = \$3$.

- Factoring/synthesis: If an assumption costs

  $$\exists \ldots x, \ldots y, \ldots (\ldots P(x)^{\$20} \wedge \ldots P(y)^{\$10} \ldots),$$

  then a “synthesis” of x and y leads to lower cost:

  $$\exists \ldots x, \ldots (\ldots P(x)^{\$10} \wedge \ldots)$$

Further developments

- Abductive syntax: The $\text{Syn}(t, e, \ldots)$ predicate expresses that the t text conveys eventuality e. Interpreting a sentence s requires proving $\exists e\ \text{Syn}(s, e, \ldots)$.
- Abductive discourse interpretation: the *coherence* of the discourse also has to be proved using axioms like $\forall w_1, w_2, e_1, e_2, e(\text{Segment}(w_1, e_1) \wedge \text{Segment}(w_2, e_2) \wedge \text{CoherenceRel}(e_1, e_2, e) \rightarrow \text{Segment}(w_1 w_2, e))$
- Formalisation of core common sense theories.
- Integration of lexical resources with wider coverage: Wordnet etc.
- Probabilistic semantics for weighted abduction (to facilitate automatic learning of weights).
- Account for the brain's implementation of the abductive interpretation mechanism.

Modelling common sense reasoning

Modus ponens

Agents know and use modus ponens:

$$\forall a, p, q, i(\text{Believe}(a, p) \wedge \text{Believe}(a, i) \wedge \text{Imply}'(i, p, q) \longrightarrow \text{Believe}(a, q))$$

General beliefs

Agents can also have genuine general beliefs. E.g. Peter's believing that whales are fishes can be formalised as

$$\text{Believe}(P, I) \wedge \text{Imply}'(I, W, F) \wedge \text{Whale}'(W, V) \wedge \text{Fish}'(F, V) \wedge \text{Iv}(V)$$

where V is an *inner variable*, subject to the axiom of universal instantiation (UI)

$$\forall p, v, y(\text{Rexist}(p) \wedge \text{Iv}(v) \longrightarrow \exists q(\text{Subst}(v, p, y, q) \wedge \text{Rexist}(q)))$$

Substitution axioms

(S1) $\forall a, b, e_1, e_2, \ldots, u_i, \ldots (\text{Subst}(a, e_1, b, e_2) \wedge \Pi'(e_1, \ldots, u_i, \ldots) \rightarrow \exists \ldots, w_i, \ldots (\Pi'(e_2, \ldots, w_i, \ldots) \wedge \ldots \text{Subst}(a, u_i, b, w_i) \wedge \ldots))$

(S2) $\forall a, b, e_1, e_2, \ldots, u_i, w_i \ldots (\text{Subst}(a, e_1, b, e_2) \wedge \Pi'(e_1, \ldots, u_i, \ldots) \rightarrow (\Pi'(e_2, \ldots, w_i, \ldots) \leftrightarrow \ldots \text{Subst}(a, u_i, b, w_i) \wedge \ldots))$

(S3) $\forall a, b, e_1, \ldots, u_i, w_i, \ldots (\ldots \text{Subst}(a, u_i, b, w_i) \wedge \ldots \Pi'(e_1, \ldots, u_i, \ldots) \rightarrow \exists e_2(\Pi'(e_2, \ldots, w_i, \ldots) \wedge \text{Subst}(a, e_1, b, e_2)))$

(S4) $\forall a, b, e_1, e_2, \ldots, u_i, w_i, \ldots (\ldots \text{Subst}(a, u_i, b, w_i) \wedge \ldots \Pi'(e_1, \ldots, u_i, \ldots) \rightarrow (\Pi'(e_2, \ldots, w_i, \ldots) \leftrightarrow \text{Subst}(a, e_1, b, e_2)))$

Substitution axioms contd

(S5) $\forall a \forall b$ Subst(a, a, b, b)

(S6) $\forall a \forall b \forall c(\neg$Eventualiy$(c) \wedge c \neq a \rightarrow$ Subst$(a, c, b, c))$

A universal instantiation example

John believes that everything is material, *therefore* John believes that Peter is material.

(i) Believe$'(e, J, u) \wedge$ Rexist$(e) \wedge$ Material$'(u, v) \wedge$ Iv(v)

(ii) by (UI), $\exists e'$: Subst$(v, e, P, e') \wedge$ Rexist(e')

(iii) by (S5) and (S6), Subst$(v, v, P, P) \wedge$ Subst(v, J, P, J)

(iv) by (S3), $\exists u'$: Subst$(v, u, P, u') \wedge$ Material$'(u', P)$

(v) by (S2), Believe$'(e', J, u)$

Logical operators

Conjunction

$$\forall e_1, e_2(\text{And}(e_1, e_2) \leftrightarrow \text{Rexist}(e_1) \wedge \text{Rexist}(e_2))$$

Implication

$$\forall e_1, e_2(\text{Imply}(e_1, e_2) \leftrightarrow (\text{Rexist}(e_1) \rightarrow \text{Rexist}(e_2)))$$

Negation

$$\forall e(\text{Not}(e) \leftrightarrow \neg\text{Rexist}(e))$$

Negation is intended to be *weak*: From

$$\Pi'(e, x_1, \ldots, x_n) \wedge \text{Not}(e)$$

it should *not* follow that $\neg\Pi(x_1, \ldots, x_n)$, because Not denies only the real existence of a particular eventuality. On the other hand, instances of the following schema hold:

$$\forall x_1, \ldots, x_n(\neg\Pi(x_1, \ldots, x_n) \leftrightarrow \forall e(\Pi'(e, x_1, \ldots, x_n) \rightarrow \text{Not}(e)))$$

Tarski's recipe for inconsistency

An $\mathcal{L}$ language is semantically closed if

(i) every sentence S of $\mathcal{L}$ has a name "S" in L

(ii) the language contains a T truth-predicate, for which all instances of the T-schema

$$S \leftrightarrow \mathrm{T}(\text{"}S\text{"})$$

are true.

If, in addition, a premise equivalent to

(iii) $S \leftrightarrow \neg\mathrm{True}(\text{"}S\text{"})$

can be established, then $\mathcal{L}$ is inconsistent.

Tarski's footnote hint: (iii) can be based on the Heterological Paradox.

The Heterological Paradox

"A predicate expression is *heterological* if and only if it doesn't apply to itself, *autological*, if it does. For example, 'is monosyllabic', 'is a French phrase' [...] are heterological since they don't apply to themselves, whereas 'is polysyllabic', 'is an English phrase' [...] are autological. Is 'is heterological' heterological? If it is heterological, it doesn't apply to itself, and so it is not. If it is not, it does apply to itself, and so is heterological. In other words, it is if and only if it isn't."

(Michael Clark, *Paradoxes from A–Z*, 2002)

A variant with substitution: If $\mathfrak{h}$ is the name of the predicate

> Substituting the free variable in x with the name of x results in a sentence which is not true.

Then the following satisfies Tarski's (iii), and paradoxical:

> Substituting the free variable in $\mathfrak{h}$ with the name of $\mathfrak{h}$ results in a sentence which is not true.

Building the Paradox in Hobbs's system

Predicates will be modelled by eventualities, and the role of the free variable will be played by an individual that is neither an inner variable nor an eventuality. Accordingly, we assume that

$$\exists x(\neg \text{Eventuality}(x) \wedge \neg \text{Iv}(x)) \quad \text{(X)}$$

We also assume the existence of four inner variables:

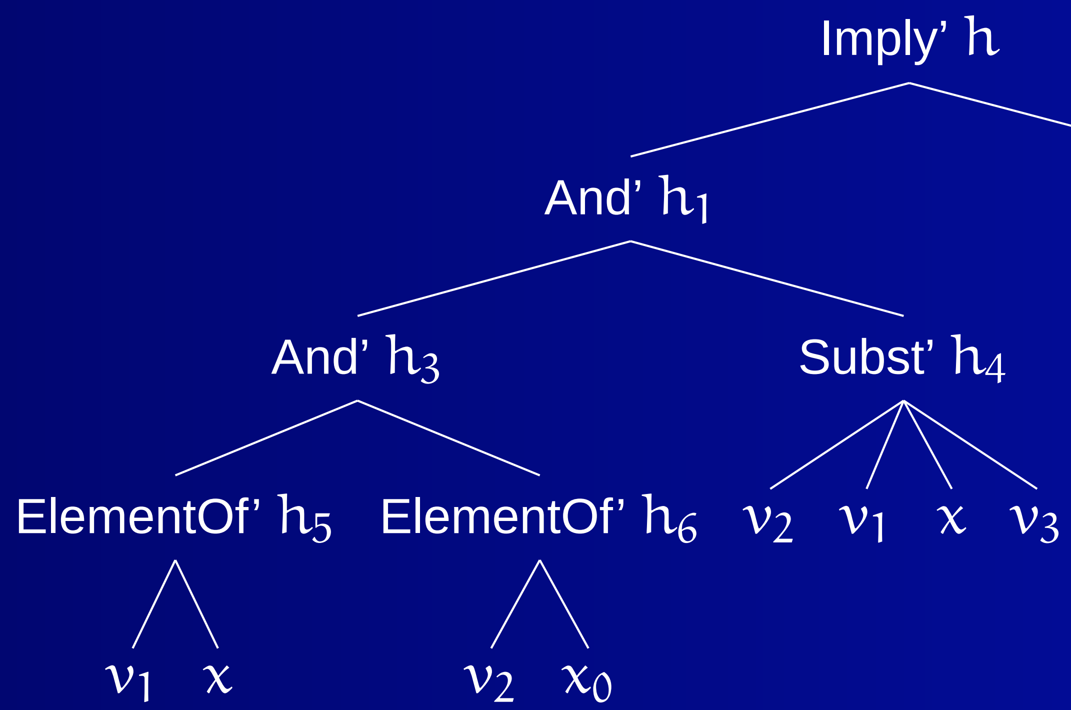
$$\exists v_1, \ldots, v_4(\text{Iv}(v_1) \wedge \ldots \text{Iv}(v_4) \wedge v_1 \neq v_2 \wedge \ldots \wedge v_3 \neq v_4) \quad \text{(IV)}$$

Finally, singletons will be used as "names" of objects, and we suppose (at least for the moment) that everything has a singleton:

$$\forall i \exists! j \forall k(\text{ElementOf}(k, j) \leftrightarrow k = i) \quad \text{(S)}$$

The "heterological" predicate

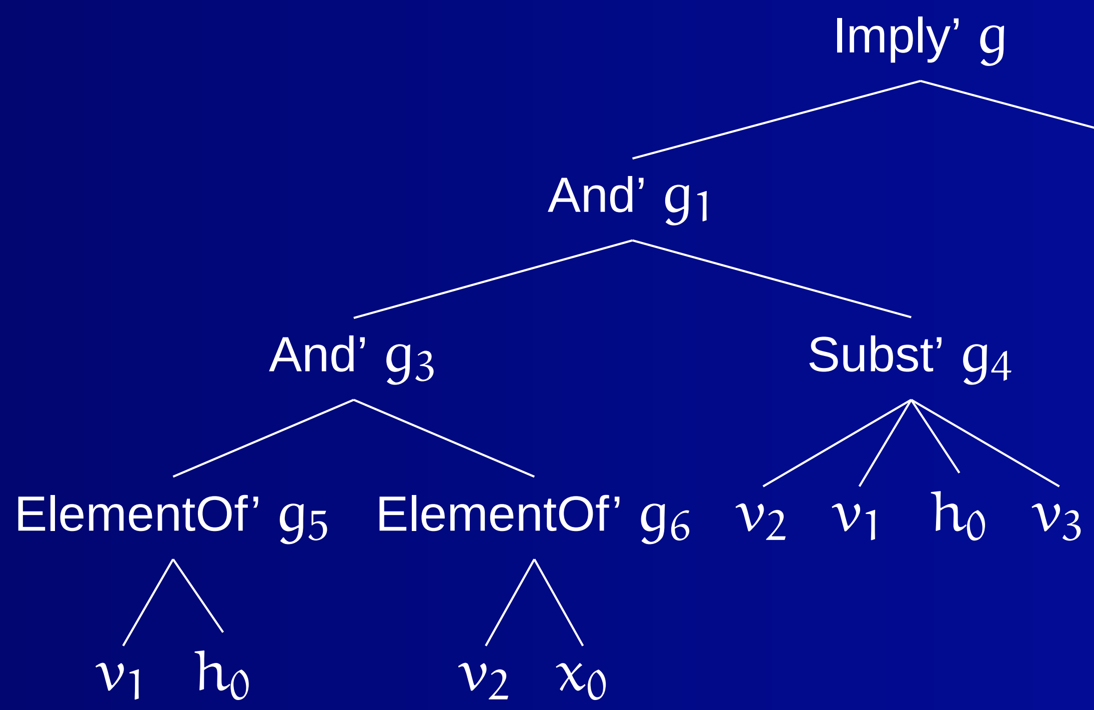
Assuming (X), (IV) and (S) it is provable from the axioms we have encountered so far that there exist an x_0 which is the singleton of x and $h, h_1, \ldots, h_6$ for which the following conditions hold:



h "says" that any eventuality that is the result of substituting its argument in the "predicate" which it "refers to" (i.e. its element) is not really existing.

"Heterological" is heterological

By our assumption regarding the existence of singletons, there will be a h_0 for which $h_0 = \{h\}$, and there will also be a g satisfying the conditions



Here g corresponds to the paradoxical " 'heterological' is heterological" statement.

“Heterological” is heterological contd

The relationship between h and g, namely

$$\mathrm{Subst}(x, h, h_0, g)$$

can be proved using the reasonable assumption that inner variables and sets (singletons) are not eventualities:

$$\forall y(\mathrm{Iv}(y) \vee \mathrm{Set}(y) \rightarrow \neg \mathrm{Eventuality}(y)) \qquad \text{(E)}$$

In that case, (S5) and (S6) guarantees that $\mathrm{Subst}(x, l, h_0, l')$ holds for all corresponding l, l' entities at the same leaves, and the bottom-up (S4) ensures that this relationship is inherited by all nodes of the tree, up to h and g at the root.

Predication trees

Definition 1 A *predication tree* is an ordered triple $T = \langle t, f, g \rangle$ where t is an ordered rooted tree with more than one nodes, f is a function mapping all non-leaf nodes of t to a primed predicate, and g is a function which maps each node of t to a term (individual constant or individual variable).

Definition 2 If $T = \langle t, f, g \rangle$ is a predication tree, and n is one of the non-leaf nodes of t, then $\mathcal{F}(n)$, the formula *belonging to* n, is the atomic formula whose predicate is $f(n)$, and its self argument is $g(n)$, while its further arguments are the terms to which g maps the children of n (in the order corresponding to the ordering of the nodes).

Predication trees contd

Definition 3 If $\mathrm{T} = \langle \mathrm{t}, \mathrm{f}, \mathrm{g} \rangle$ is a predication tree, then $\mathcal{F}(\mathrm{T})$, the formula *belonging to* T, is the conjunction of all atomic formulas that belong to the non-leaf nodes of t (in the order corresponding to the ordering of the nodes).

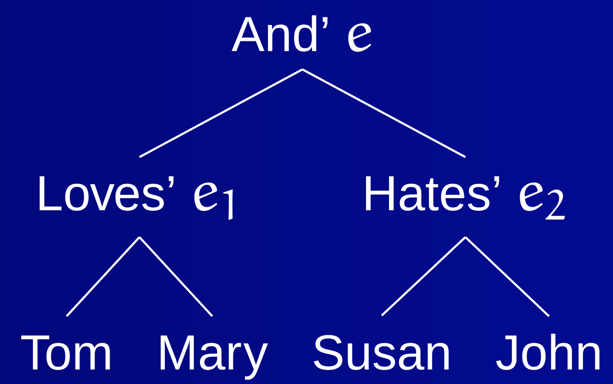
Definition 4 If $\mathrm{T} = \langle \mathrm{t}, \mathrm{f}, \mathrm{g} \rangle$ is a predication tree, then $\mathcal{C}(\mathrm{T})$, the *completeness formula* of T, is the conjunction containing, for each l leaf of t, a conjunct of the form

$$\ulcorner \neg \text{Eventuality}(\tau) \urcorner$$

where $\tau = \mathrm{g}(\mathrm{l})$, and the order of the conjuncts follows the ordering of the leafs.

Predication trees: an example

For instance, if T is the predication tree



then $\mathcal{F}(\mathrm{T})$ is the formula

$$\text{And}'(e, e_1, e_2) \wedge \text{Loves}'(e_1, \text{Tom}, \text{Mary}) \wedge \text{Hates}'(e_2, \text{Susan}, \text{John})$$

while $\mathcal{C}(\mathrm{T})$ is the formula

$$\neg\text{Eventuality}(\text{Tom}) \wedge \neg\text{Eventuality}(\text{Mary}) \wedge \neg\text{Eventuality}(\text{Susan}) \wedge \neg\text{Eventuality}(\text{John})$$

Predication trees contd

Proposition 1 If $T_1 = \langle t, f, g_1 \rangle$, $T_2 = \langle t, f, g_2 \rangle$ are predication trees, the root of t is r, α and β are terms and φ_l is a conjunction containing for each l leaf of t a conjunct

$$\ulcorner \text{Subst}(\alpha, g_1(l), \beta, g_2(l)) \urcorner$$

while φ_t is a conjunction containing for each n node of t a conjunct

$$\ulcorner \text{Subst}(\alpha, g_1(n), \beta, g_2(n)) \urcorner$$

then the following formulas are provable from the substitution axioms:

(a) $\ulcorner \mathcal{F}(T_1) \wedge \mathcal{F}(T_2) \wedge \text{Subst}(\alpha, g_1(r), \beta, g_2(r)) \longrightarrow \varphi_t \urcorner$

(b) $\ulcorner \mathcal{F}(T_1) \wedge \mathcal{F}(T_2) \wedge \varphi_l \longrightarrow \varphi_t \urcorner$

(c) $\ulcorner \mathcal{F}(T_1) \wedge \mathcal{F}(T_2) \longrightarrow (\text{Subst}(\alpha, g_1(r), \beta, g_2(r)) \leftrightarrow \varphi_l) \urcorner$

Predication trees contd

Proposition 2 If $T_1 = \langle t, f, g_1 \rangle$ is a predication tree and the nodes of t are $n_0, \ldots n_m$ with n_0 as root, α, β, τ are terms, $\gamma_1, \ldots, \gamma_m$ are different variables also different from τ, and $T_2 = \langle t, f, g_2 \rangle$ is a predication tree for which $g_2(t_0) = \tau$, and $g_2(t_i) = \gamma_i$ for all $i \in [1, \ldots, m]$, then it is provable from the substitution axioms that

$$\ulcorner \mathcal{F}(T_1) \wedge \text{Subst}(\alpha, g_1(n_0), \beta, \tau) \rightarrow \exists \gamma_1, \ldots \gamma_m (\mathcal{F}(T_2) \wedge \text{Subst}(\alpha, g_1(n_1), \beta, \gamma_1) \wedge \ldots \wedge \text{Subst}(\alpha, g_1(n_m), \beta, \gamma_m))\urcorner$$

Provable using (S3), by induction on the number of nodes in the tree.

Predication trees contd

Surprisingly, eventualities resulting from the same eventualities by the same substitution have the same atomic properties and relations:

Theorem 1 The following is provable from Hobbs's axioms for any n-ary Π predicate: If $e_1, \ldots e_n, d_1, \ldots, d_n, d'_1, \ldots, d'_n$ and a, b are eventualities for which

$\mathrm{Subst}(a, e_1, b, d_1) \wedge \mathrm{Subst}(a, e_1, b, d'_1) \wedge \ldots \wedge \mathrm{Subst}(a, e_n, b, d_n) \wedge \mathrm{Subst}(a, e_n, b, d'_n)$

then

$$\Pi(d_1, \ldots, d_n) \leftrightarrow \Pi(d'_1, \ldots, d'_n)$$

Predication trees contd

Proof sketch. If $\Pi(d_1, \ldots, d_n)$ holds, then it will also hold that

$$\exists e(\text{Rexist}(e) \wedge \Pi'(e, d_1, \ldots, d_n)).$$

Let f be an eventuality for which

$$\Pi'(f, e_1, \ldots, e_n).$$

On the basis of axiom schema (S4) and our assumptions we can infer that

$$\text{Subst}(a, f, b, e)$$

holds. From this, applying (S4) again we also get

$$\Pi'(e, d_1', \ldots, d_n')$$

from which, considering that $\text{Rexist}(e)$, it follows that $\Pi(d_1', \ldots, d_2')$.

Predication trees contd

A trivial, but important consequence of the previous theorem:

$$\text{Subst}(a, e, b, d) \wedge \text{Subst}(a, e, b, d') \rightarrow (\text{Rexist}(d) \leftrightarrow \text{Rexist}(d'))$$

It is also provable that isomorphic eventualities, that is, eventualities involving the same non-eventualities and the same predicate structure also have the same atomic properties.

Definition 5 If τ_1 and τ_2 are terms and φ is a formula, then φ is an *isomorphism formula* between τ_1 and τ_2, if there is a tree t and there are mappings f, g_1, g_2 such that both $T_1 = \langle t, f, g_1 \rangle$ and $T_2 = \langle t, f, g_2 \rangle$ are predication trees, for every l leaf of t $g_1(l) = g_2(l)$, g_1 maps the root of t to τ_1, g_2 maps the root of t to τ_2, and φ is a conjunction consisting of the following conjuncts: $\mathcal{F}(T_1), \mathcal{C}(T_1), \mathcal{F}(T_2)$.

Predication trees contd

Lemma 1 If τ_1 and τ_2 are terms, φ is an isomorphism formula between them, then for any term α the following is provable from Hobbs's axioms:

$$\ulcorner \varphi \rightarrow \text{Subst}(\alpha, \tau_1, \alpha, \tau_2) \urcorner$$

Using that lemma it is provable that isomorphic eventualities have the same atomic properties:

Theorem 2 If Π is an atomic predicate with arity n, $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are terms, and $\varphi_1, \ldots, \varphi_n$ are formulas such that for all $i \in [1, n]$, φ_i is an ismorphism formula between α_i and β_i, then it is provable from the axioms that

$$\ulcorner \varphi_1 \wedge \ldots \wedge \varphi_n \rightarrow (\Pi(\alpha_1, \ldots, \alpha_n) \leftrightarrow \Pi(\beta_1, \ldots, \beta_n)) \urcorner$$

The collapse of grounded eventualities

Applying Theorem 2 to Rexist, we get that if φ is an isomorphism formula between α and β, then

$$\ulcorner \varphi \longrightarrow (\text{Rexist}(\alpha) \leftrightarrow \text{Rexist}(\beta)) \urcorner$$

is provable from the axioms. This means that the axiom system is not as Davidsonian as it was intended to be. E.g., from the assumption that $\neg\text{Eventuality}(\text{John}) \wedge \text{Runs}'(e_1, \text{John}) \wedge \neg\text{Rexist}(e_1)$ it is provable that

$$\forall e(\text{Runs}'(e, \text{John}) \longrightarrow \neg\text{Rexist}(e))$$

from which it follows that $\neg\text{Runs}(\text{John})$, i.e. denying a particular condition of John's running, we deny all such conditions.

Perhaps even more dramatically, if the predicate Past applies to one eventuality of John's running, then it applies to all such eventualities.

Reconstructing the paradoxical reasoning

To prove that

$$\text{Rexist}(g) \longrightarrow \neg\text{Rexist}(g)$$

we assume that

$$\forall y(\text{Iv}(y) \longrightarrow \neg\text{Set}(y)) \text{ and } \forall y(\text{Iv}(y) \vee \text{Set}(y) \longrightarrow \neg\text{Eventuality}(y))$$

and observe that the already proved

$$\text{Subst}(a, e, b, d) \wedge \text{Subst}(a, e, b, d') \longrightarrow (\text{Rexist}(d) \leftrightarrow \text{Rexist}(d'))$$

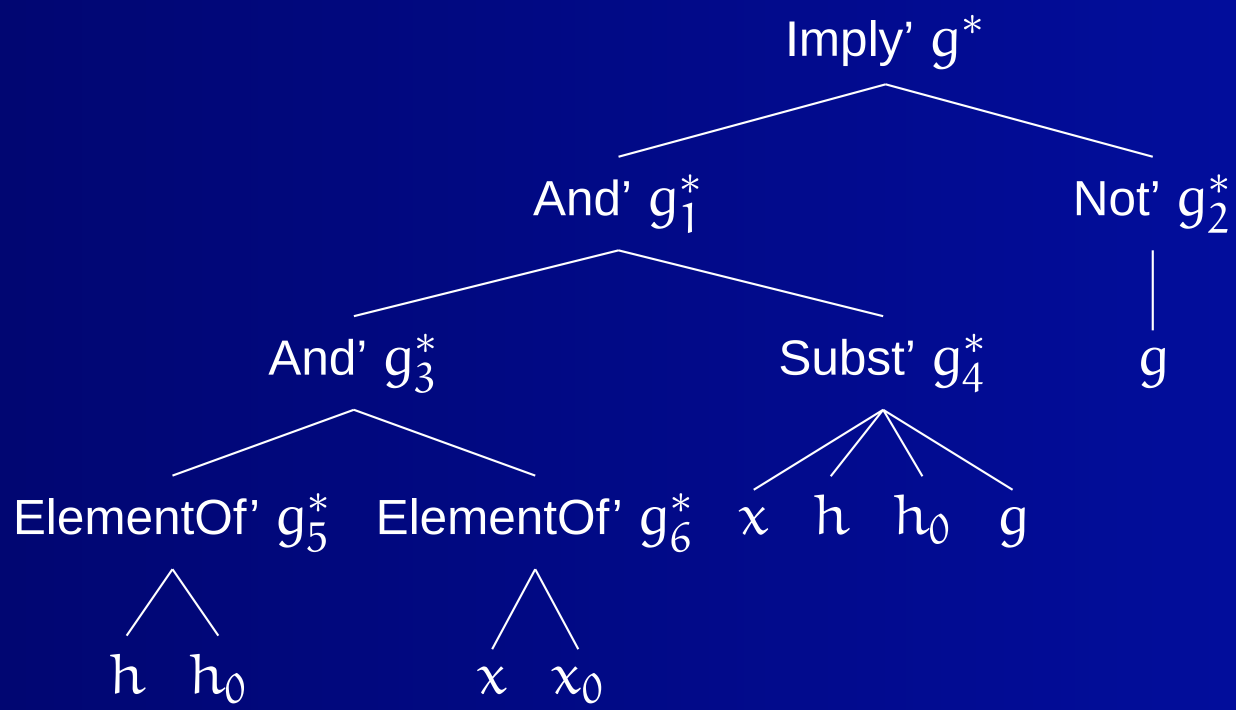
can be used to strengthen (UI) to the form

$$\forall v, y, p(\text{Rexist}(p) \wedge \text{Iv}(v) \longrightarrow$$
$$(\exists q.\text{Subst}(v, p, y, q) \wedge \forall r(\text{Subst}(v, p, y, r) \longrightarrow \text{Rexist}(r))))$$

and on the basis of this variant of (UI), we can perform a series of “Rexist preserving” substitutions in g, replacing v_1 with h, v_2 with x, and v_3 with g itself.

Reconstructing the paradoxical reasoning contd

For the resulting g^* both Rexist(g^*) and the formula belonging to the following tree is provable:



From which it can be computed that Rexist(g_2^*) holds, and, therefore, Not$(g) \leftrightarrow \neg$Rexist(g) must also hold.

Reconstructing the paradoxical reasoning contd

The proof of the other direction:

$$\neg \text{Rexist}(g) \longrightarrow \text{Rexist}(g)$$

requires assuming the following (R) reverse version of (UI):

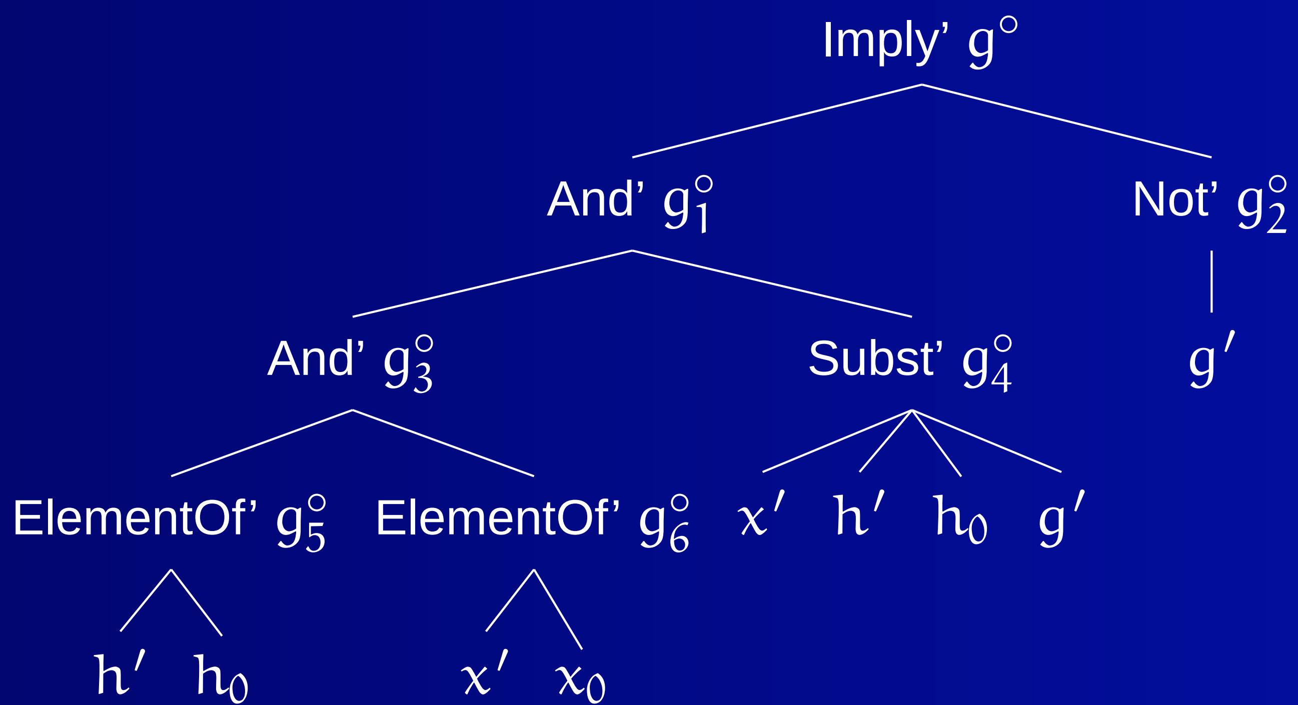
$$\forall v, p(\neg \text{Rexist}(p) \wedge \text{Iv}(v) \longrightarrow$$
$$\exists y, q(\neg \text{Iv}(y) \wedge \text{Subst}(v, p, y, q) \wedge \neg \text{Rexist}(q)))$$

which states that not "Rexisting" universal eventualities must have counterexamples.

Assuming (R), we can perform a series of "falsity preserving" substitutions in g, replacing v_1 with h', v_2 with x', and v_3 with g' to arrive at the result g°, for which $\neg \text{Rexist}(g^\circ)$ is provable.

Reconstructing the paradoxical reasoning contd

The formula belonging to the following tree is provable:



A series of indirect arguments shows that Rexist(g_1°), Rexist(g_3°), $\ldots$, Rexist(g_6°) and Rexist(g') hold. Therefore $h' = h$, $x' = x$, and Subst(x, h, h_0, g'), from which g and g' are isomorphic and therefore Rexist(g) must hold.

Conclusions

- The “collapse of grounded eventualities” is provable from the axioms about substitution and Rexist.
- The $\text{Rexist}(g) \longrightarrow \neg\text{Rexist}(g)$ direction of the paradox requires assuming the existential (X), (IV), plus that inner variables and sets are not eventualities, and that no inner variable is a set.
- The $\neg\text{Rexist}(g) \longrightarrow \text{Rexist}(g)$ direction also requires the reverse version of (UI).
- All assumptions except the reverse of (UI) might still hold in sufficiently strange models.